UDC 534

ON PERIODIC SOLUTIONS CLOSE TO RECTILINEAR NORMAL VIBRATION MODES

PMM Vol. 36, №6, 1972, pp. 1051-1058

L. I. MANEVICH and Iu. V. MIKHLIN
(Dnepropetrovsk)

(Received September 28, 1971)

Periodic solutions of essentially nonlinear systems similar to normal vibrations with rectilinear trajectories are investigated. Systems with homogeneous potentials are assumed to be the generating systems. Normal vibration modes of some nonlinear conservative systems with a finite number of degrees of freedom, which are an extension of normal vibrations of linear systems have been studied in several papers in recent years [1, 2]. Rectilinear trajectories in configuration space correspond to the known exact solutions of the normal vibration problems. These solutions can be used as generators in determining the periodic motions of systems similar to those studied. The existence of solutions close to linear normal vibrations has been proved in the Liapunov works [3, 4] for a broad class of quasi-linear systems. Qualitative questions of the theory of normal vibrations with curvilinear trajectories, as well as an approximate construction of normal vibrations in several particular cases, have been considered in [5 - 7].

1. Let us consider a conservative system described by the differential equations

$$x_s^{\cdot\cdot} = f_s(x_1, x_2, \ldots, x_n) \qquad (s = 1, 2, \ldots, n) \tag{1.1}$$

where f_s are odd analytic functions of $x_1, x_2, \ldots, x_n$ in a closed domain of configuration space. Keeping only terms of the least, rth power in $x_1, x_2, \ldots, x_n$ in (1.1), we obtain the generating homogeneous system associated with (1.1). The normal mode vibrations of a homogeneous system are determined by the relationships $x_{s0} = C_s x_n$ $(s = 1,2,\ldots, n-1)$, where the constants C_s are found from the algebraic equations [1]

$$C_s f_n^{(r)}(C_1, C_2, \ldots, C_{n-1}, 1) = f_s^{(r)}(C_1, C_2, \ldots, C_{n-1}, 1) \quad (s = 1, 2, \ldots, n-1) \tag{1.2}$$

Here and henceforth, $f_n^{(r)}$, $f_s^{(r)}$ are components of lowest order in $x_1, x_2, \ldots, x_n$ in the expansions of the functions f_n, f_s. Without limiting the generality, let us assume that a system of coordinates in which $C_s = 0$, has been selected, which means that also $x_{s0} = 0$ $(s = 1,2,\ldots, n-1)$. Let us introduce the notation $x_n = x$, $f_n = f$ in this coordinate system.

In order to construct periodic solutions of (1.1) close to the normal vibrations of a homogeneous system, let us initially determine the trajectories of the desired periodic solutions $x_s = x_s(x)$ $(s = 1,2,\ldots, n-1)$. The equations to determine the trajectories can be written as follows:

$$2\left[h - F(x, x_1(x), x_2(x), \ldots, x_{n-1}(x)\right]\left[1 + \sum_{i=1}^{n-1}\left(\frac{dx_i}{dx}\right)^2\right]^{-1}\frac{d^2x_s}{dx^2} + f(x, x_1(x), \ldots, x_{n-1}(x))\frac{dx_s}{dx} = f_s(x, x_1(x), \ldots, x_{n-1}(x)) \tag{1.3}$$

$$(s = 1, 2, \ldots, n-1)$$

where h is a constant energy, and F is the potential of the system (1.1). Let us take the solution of (1.3) in the form of the series

$$x_s = \sum_{k=0}^{\infty} x_{sk}(x) \qquad (s = 1, 2, \ldots, n-1) \tag{1.4}$$

In the zero-th approximation $x_{s0} = 0$. As in rectilinear normal trajectories, the trajectory (1.4) should satisfy the boundary conditions

a) $x_s(0) = 0 \qquad (s = 1, 2, \ldots, n-1)$

b) On the maximum isoenergetic surface

$$F(X, x_1(X), x_2(X), \ldots, x_{n-1}(X)) = h \tag{1.5}$$

orthogonality conditions for the trajectories to this surface must be satisfied

$$\left.\frac{dx_s}{dx}\right|_{x=X} f(X, x_1(X), \ldots, x_{n-1}(X)) = f_s(X, x_1(X), \ldots, x_{n-1}(X)) \tag{1.6}$$

where X is the amplitude value of the variable x. After $x_s(x)$ has been determined, the problem reduces to integrating the equation

$$x^{\cdot\cdot} = f(x, x_1(x), x_2(x), \ldots, x_{n-1}(x)) \tag{1.7}$$

The proof of the existence and the construction of a unique periodic solution are carried out under the following constraints on the system (1.1):

1) The determinants are $\qquad \Delta_m \neq 0 \qquad$ (1.8)

$$\Delta_m = \left| \delta_s^{\,j} m(m-1) \frac{2f^{(r)}(1, 0, \ldots, 0)}{r+1} + \delta_s^{\,j} m f^{(r)}(1, 0, \ldots, 0) - \frac{\partial f_s^{(r)}}{\partial x_j}(1, 0, \ldots, 0) \right|$$

where $\delta_s^{\,j}$ are the Kronecker deltas, $m = 1, 2, \ldots$

2) Equilibrium positions are absent on the maximum isoenergetic surface.

If $r = 1$, then the constraint (1.8) agrees with the condition excluding multiple frequencies in the generating system, which Liapunov took in the investigation of quasilinear systems [5].

2. Let us turn to the construction of an asymptotic process which will afford the possibility of determining the system trajectory. Let us take the solution of the homogeneous system $x_{s0} = 0$ as the zero-th approximation.

Let the functions $x_{sm}(x)$ $(m < k)$ be defined. We then obtain the following kth approximation equation

$$\begin{gathered} 2\frac{d^2 x_{sk}}{dx^2}\left[h + \frac{f^{(r)}(x, 0, \ldots, 0)}{r+1} x\right] + \frac{dx_{sk}}{dx} f^{(r)}(x, 0, \ldots, 0) - \\ \sum_{j=1}^{n-1} \frac{\partial f_s^{(r)}(x, 0, \ldots, 0)}{\partial x_j} x_{jk} + \sum_{l=1}^{k-1} \frac{d^2 x_{sl}}{dx^2} N_{k-l}^{(1)} + \sum_{l=1}^{k-1} \frac{dx_{sl}}{dx} N_{k-l}^{(2)} - \\ \left[N_k^{(3)} - \sum_{j=1}^{n-1} \frac{\partial f_s^{(r)}}{\partial x_j}(x, 0, \ldots, 0) x_{jk}\right] = 0 \end{gathered} \tag{2.1}$$

Here

$$N_m^{(i)} = \sum C^{(\gamma)} \frac{\partial^\gamma N^{(i)}(x, 0, \ldots, 0)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_{n-1}^{\alpha_{n-1}}} \prod_{j=1}^{n-1} \prod_{q=1}^{m} (x_{jq})^{\beta_q^{(j)}}$$

The sign Σ is extended over all solutions in positive integers of the equation

$$\sum_{j=1}^{n-1} (\beta_1^{(j)} + 2\beta_2^{(j)} + \ldots + m\beta_m^{(j)}) = m$$

Hence

$$\sum_{q=1}^{m} \beta_q^{(j)} = \alpha_j \quad \sum_{j=1}^{n-1} \alpha_j = \gamma \quad \sum_{j=1}^{n-1} \beta_q^{(j)} = \xi_q$$

$$C^{(\gamma)} = m! \left[\prod_{q=1}^{m} \xi_q! \, (q!)^{\xi_q}\right]^{-1}$$

$$N^{(1)} = -2F(x, x_1(x), x_2(x), \ldots, x_{n-1}(x))$$

$$N^{(2)} = f(x, x_1(x), x_2(x), \ldots, x_{n-1}(x)) \left[1 + \sum_{j=1}^{n-1} \left(\frac{dx_j(x)}{dx}\right)^2\right]$$

$$N^{(3)} = f_s(x, x_1(x), x_2(x), \ldots, x_{n-1}(x)) \left[1 + \sum_{j=1}^{n-1} \left(\frac{dx_j(x)}{dx}\right)^2\right]$$

The solution (2.1) is sought in the form

$$x_{sk} = \sum_{j=1}^{\infty} A_{sj}^{(k)} x^j \tag{2.2}$$

The coefficients $A_{sj}^{(n)}$ are related by an infinite system of linear recursion relations

$$2h(r+l+2)(r+l+1) A_{s\,r+l+2}^{(k)} + l(l+1) \frac{2}{r+1} f^{(r)}(1, 0, \ldots, 0) A_{sl+1}^{(k)} +$$

$$(l+1) f^{(r)}(1, 0, \ldots, 0) A_{sl+1}^{(k)} - \sum_{j=1}^{n-1} \frac{\partial f_s^{(r)}}{\partial x_j}(1, 0, \ldots, 0) A_{jl+1}^{(k)} = \Phi_l^{(k)} \tag{2.3}$$

where the function $\Phi_l^{(k)}$ depends on the preceding approximations. If conditions (1.8) are satisfied, then all the coefficients of the series (2.2) are expressed uniquely in terms of the $n-1$ quantities $A_{jp}^{(k)}$ ($j = 1,2,\ldots, n-1$; p is any fixed integer).

The boundary conditions (1.6) corresponding to the kth approximation

$$\frac{dx_{sk}}{dx}\bigg|_{x=X} f^{(r)}(X, 0, \ldots, 0) - \sum_{j=1}^{n-1} \frac{\partial f_s^{(r)}}{\partial x_j}(X, 0, \ldots, 0)\, x_{jk}(X) + \tag{2.4}$$

$$\sum_{l=0}^{k-1} \frac{dx_{sl}}{dx}\bigg|_{x=X} N_{k-l}^{(2)}\Big|_{x=X} - \left[N_k^{(3)}\Big|_{x=X} - \sum_{j=1}^{n-1} \frac{\partial f_s^{(r)}}{\partial x_j}(X, 0, \ldots, 0)\, x_{jk}(X)\right] = 0$$

should be used to determine $A_{jp}^{(k)}$. Because of the constraint (1.8), necessary and sufficient conditions for the coefficients $A_{jp}^{(k)}$ to be represented uniquely as power series in X from (2.4), are satisfied.

Taking account of all the approximations, we obtain the trajectory (1.4) which depends on the parameter X. Later, the convergence of the series (2.2) and (1.4) in some neighborhood of the origin will be proved. For a specified energy level of the system, the "amplitude" of the vibrations X can be determined from (1.5) as an analytic function of the energy h. After the trajectory has been constructed, the solution of (1.7) can be obtained in quadratures. Since the root of (1.5) is simple, as follows from the proposition (2), the motion of the conservative system with one degree of freedom (1.7) is periodic in the domain $F(x) \leqslant h$.

3. Let us prove the convergence of the series obtained formally. Let us consider first the series (2.2) by assuming that the boundedness of $x_{sm}(x)$ has been proved for $m < k$. (The boundedness of $x_{s0}(x) = C_s x$ for finite values of x is evident). Furthermore, let the coefficients of (2.2) be replaced everywhere by their absolute values, and also replace x by $|X|$.

Since F, f, f_s are analytic functions of $x, x_1, x_2, \ldots, x_{n-1}$, the estimates

$$\left\{\frac{C^{(\gamma)}}{k}\left|\frac{\partial^\gamma N^{(i)}(X,0,\ldots,0)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_{n-1}^{\alpha_{n-1}}}\right|,\ \left|f^{(r)}(X,0,\ldots,0)\right|,\ \left|\frac{\partial f_s^{(r)}}{\partial x_j}(X,0,\ldots,0)\right|\right\} < B$$

$$(0 < B < \infty) \qquad (3.1)$$

are then valid in the domain of definition of (1.1).

Let us examine the set of coefficients $A_{sj}^{(k)}$ $(s = 1,2,\ldots,\ n-1,\ j \leqslant l)$. We suppose that these coefficients are bounded. Then without limiting the generality, it can be assumed that the following inequalities are satisfied:

$$|A_{sj}^{(k)}| < \left(\frac{a_l}{|X|}\right)^{j-1} A \qquad (0 < A < \infty,\ 0 < a_l < 1)$$

where A is an arbitrary, but finite, quantity exceeding the modulus of the greatest coefficient in absolute value among all the $A_{sj}^{(k)}$. Because of the arbitrariness in selecting A the quantity a_l can be made arbitrarily small. Let $M = \max\{A, B\}$. Taking account of the estimates (3.1), we obtain the following inequality from (2.3):

$$\frac{|A_{sl+1}^{(k)}|}{|A_{sl}^{(k)}|} \leqslant \frac{a_{l+1}}{|X|}$$

Here

$$a_{l+1} = \max\left\{a_l,\ a_l\frac{l-1}{l+1}\,\frac{[(l-r)(2l-2r-1)+(n-1)+\Phi_{l+1}^{(k)}/M^2(|X|/a_l)^l]}{[(l-r-1)(2l-2r-4)+(n-1)+\Phi_l^{(k)}/M^2(|X|/a_l)^{l-1}]}\right\}$$

Since $a_{l+1} > a_l$, the quantities $A_{sj}^{(k)}$ $(j \leqslant l+1)$ satisfy the inequality

$$|A_{sj}^{(k)}| < \left(\frac{a_{l+1}}{|X|}\right)^{j-1} M$$

Let us introduce the constants $a_{l+2}, a_{l+3}, \ldots$ analogously. Then $|A_{sj}^{(k)}| < \left(\frac{a_q}{|X|}\right)^{j-1} M$ ($j \leqslant q$ and q increases without limit). Let us consider the limit value of a_q

$$\lim_{q\to\infty} a_q = a_l \prod_{q=l}^{\infty}\left\{\frac{q-1}{q+1}\cdot\frac{[(q-r)(2q-2r-1)+(n-1)+\Phi_{q+1}^{(k)}/M^2(|X|/a_q)^q]}{[(q-r-1)(2q-2r-4)+(n-1)+\Phi_q^{(k)}/M^2(|X|/a_q)^{q-1}]}\right\} \qquad (3.2)$$

If the functions $x_{sm}(x)$ are bounded for $m < k$, then the infinite product (3.2) converges. Making the quantity a_l sufficiently small, we obtain $a = \lim_{q\to\infty} a_q < 1$, and the menbers of the series (2.2) decrease in a geometric progression.

It is also necessary to prove that the coefficients $A_{sj}^{(k)}$ $(j \leqslant l)$ are bounded. It follows from (2.3) that all $A_{sj}^{(\eta)}$ are linear functions of the greatest coefficients $A_{s\eta}^{(k)}$ ($\eta \leqslant l$, η is fixed) in absolute value

$$A_{sj}^{(k)} = \sum_{i=1}^{n-1} u_{si}A_{i\eta} + v_s \qquad (s = 1, 2, \ldots, n-1)$$

where u_{si} and v_s are bounded quantities. Therefore, (2.4) connects the analytic functions of X and $A_{s\eta}$, and a value $X = X_0$ exists such that quantities $A_{s\eta}$ are represented by power series in X for $|X| < |X_0|$ [8]. The series (2.2) converge for these values of X.

Now, let us prove that the coefficients of the series (1.4) decrease in a geometric progression as the number of the approximation k grows. Suppose that the decrease of the coefficients $A_{sj}^{(k)}$ with increasing k for $j \leqslant m$, is proved, and

$$\{j(j-1)\,|A_{sj}^{(k)}|,\ |A_{sj}^{(k)}|\} < (b_l)^{k-1} P \qquad (0 < P < \infty,\ 0 < b_l < 1) \tag{3.3}$$

Examining the relations (2.3) and (2.4) for different k, it can be shown that the inequalities are valid, particularly for $A_{s1}^{(k)}$. Let $K = \max\{P, B\}$. Using the estimates (3.1) and (3.3) it can be shown that

$$N_p^{(i)} < p\,(b_l)^{p-1} R \qquad (K < R < \infty) \tag{3.4}$$

It has been proved earlier that the coefficients $A_{sm+1}^{(k)}$ are bounded. Hence, values of the constants b_l and R can be selected such that the conditions

$$(m+1)\,m\,|A_{sm+1}^{(k)}| < (b_l)^{k-1} R, \qquad k \leqslant l$$

would be satisfied, where b_l can be made arbitrarily small if R is sufficiently large. Taking account of (3.1), (3.3), (3.4), we obtain the following inequality from (2.3):

$$\frac{m\,(m+1)\,|A_{sm+1}^{(l+1)}|}{m\,(m+1)\,|A_{sm+1}^{(l)}|} < b_{l+1}$$

$$b_{l+1} = \max\left\{b_l,\ b_l\,\frac{2\,(m+2)\,(l+1)\,(l+2)\,(2l+3) + 3\,(l+2)^2}{2\,(m+2)\,l\,(l+1)\,(2l+1) + 3\,(l+1)^2}\right\}$$

Let us also introduce the constants $b_{l+2}, b_{l+3}, \ldots$. We examine

$$b = \lim_{q\to\infty} b_q = b_l \prod_{q=l}^{\infty}\left[\frac{2\,(m+2)\,(q+1)\,(q+2)\,(2q+3) + 3\,(q+2)^2}{2\,(m+2)\,(q+1)\,q\,(2q+1) + 3\,(q+1)^2}\right]$$

The infinite products converge and we obtain the inequality $b < 1$ by selecting the quantities b_l sufficiently small. It hence follows that the coefficients $A_{sm+1}^{(k)}$ decrease in a geometric progression as the number of the approximation k increases.

The proof that the quantities $A_{sj}^{(k)}$ $(j = m+2,\ m+3, \ldots)$ decrease in a geometric progression is carried out analogously. Finally, it follows from the construction of the asymptotic process that the series (1.4) converge to the solution of (1.3). Therefore, upon compliance with the constraints (1) and (2) a unique periodic solution of (1.1) which possesses the properties of normal vibrations corresponds to each normal solution of the generating homogeneous system.

4. To illustrate the method proposed, let us examine the free vibrations of a clamped filament with two lumped masses m_1 and m_2; gravity forces are not taken into account (Fig. 1). Let l_1, l_2, l_3 be the lengths of the undeformed sections of the filament, EF the tensile stiffness of the filament, z_1, z_2 the transverse displacements of the masses m_1 and m_2.

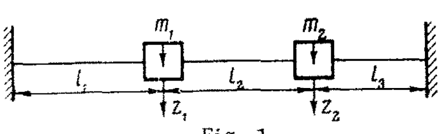

Fig. 1

Let us compute the normal transverse vibrations of the filament by assuming that there is no preliminary tension. In this case the system is essentially nonlinear and even non-linearizable, and the equations of motion are

$$m_i z_i^{\cdot\cdot} + \frac{\partial \Pi}{\partial z_i} = 0 \qquad (i = 1, 2)$$

$$\Pi = \left\{ l_1 \left[1 - \left(\frac{z_1}{l_1}\right)^2\right] \left[1 - \frac{2}{3} \sqrt{1 - \left(\frac{z_1}{l_1}\right)^2}\right] + l_2 \left[1 - \left(\frac{l_1}{l_2}\right)^2 \left(\frac{z_1}{l_1} - \frac{z_2}{l_1}\right)^2\right] \times \right.$$
$$\left[1 - \frac{2}{3} \sqrt{1 - \left(\frac{l_1}{l_2}\right)^2 \left(\frac{z_1}{l_1} - \frac{z_2}{l_1}\right)^2}\right] + l_3 \left[1 - \left(\frac{l_1}{l_3}\right)^2 \left(\frac{z_2}{l_1}\right)^2\right] \times$$
$$\left. \left[1 - \frac{2}{3} \sqrt{1 - \left(\frac{l_1}{l_3}\right)^2 \left(\frac{z_2}{l_1}\right)^2}\right] \right\}$$

Let us introduce the variables

$$x = \frac{z_1}{l_1}, \qquad y = \frac{z_2}{l_1}, \qquad \tau = \left(\frac{EF}{2m_1 l_1}\right)^{1/2} t$$

Assuming that $|x| < 1$, $|y| < l_3/l_1$, let us keep only terms containing the fourth and sixth degrees in the expansion of the potential in powers of x and y Then the equations of motion are written as follows in the dimensionless form corresponding to (1.1):

$$x^{\cdot\cdot} + x^3 + \left(\frac{l_1}{l_2}\right)^3 (x - y)^3 + \frac{1}{4} x^5 + \frac{1}{4} \left(\frac{l_1}{l_2}\right)^5 (x - y)^5 = 0 \tag{4.1}$$
$$y^{\cdot\cdot} + \frac{m_1}{m_2} \left[\left(\frac{l_1}{l_3}\right)^3 y^3 + \left(\frac{l_1}{l_2}\right)^3 (y - x)^3 + \frac{1}{4} \left(\frac{l_1}{l_3}\right)^5 y^5 + \frac{1}{4} \left(\frac{l_1}{l_2}\right)^5 (y - x)^5\right] = 0$$

Let us determine the periodic solutions of (4.1) which are close to the normal vibrations of the generating homogeneous system whose potential contains only fourth powers of x, y. The relations (1.2) to determine the vibrations mode $y_0 = Cx$ of the homogeneous system reduce to the algebraic equation

$$C \left[1 + \left(\frac{l_1}{l_2}\right)^3 (1 - C)^3\right] - \frac{m_1}{m_2} \left[\left(\frac{l_1}{l_3}\right)^3 C^3 + \left(\frac{l_1}{l_2}\right)^3 (C - 1)^3\right] = 0 \tag{4.2}$$

A rectilinear normal vibrations mode of the generating system corresponds to each real solution of (4.2). In order to construct trajectories of the periodic motions of the system (4.1) which are close to rectilinear normal modes, let us use the first approximation equation in the form (2.1). In the case under consideration we obtain

$$P_1(x)\, y_1'' + P_2(x)\, y_1' + P_3(x)\, y_1 + P_4(x) = 0 \tag{4.3}$$

$$P_1(x) = \frac{1}{2(1 + m_2/m_1 C^2)} \left[1 + \left(\frac{l_1}{l_3}\right)^3 (1 - C)^4 + \left(\frac{l_1}{l_3}\right)^3 C^4\right] (X^4 - x^4)$$

$$P_2(x) = -\left[1 + \left(\frac{l_1}{l_2}\right)^3 (1 - C)^3\right] x^3$$

$$P_3(x) = \left\{3C \left(\frac{l_1}{l_2}\right)^3 (1 - C)^2 + 3\, \frac{m_1}{m_2} \left[\left(\frac{l_1}{l_3}\right)^3 C^2 + \left(\frac{l_1}{l_2}\right)^3 (C - 1)^2\right]\right\} x^2$$

$$P_4(x) = \frac{1}{4} \left\{-C \left[1 + \left(\frac{l_1}{l_2}\right)^5 (1 - C)^5\right] + \frac{m_1}{m_2} \left[\left(\frac{l_1}{l_3}\right)^5 C^5 + \left(\frac{l_1}{l_2}\right)^5 (C - 1)^5\right]\right\} x^5$$

We represent the solution (4.3) in the form (2.2) by satisfying the boundary conditions (1.6). In a numerical computation we assume

$$\frac{l_1}{l_2} = 1, \quad \left(\frac{l_1}{l_3}\right)^3 = \frac{1}{2}, \quad \frac{m_1}{m_2} = 1, \quad x(0) = X = 0.9 \quad x^{\cdot}(0) = 0$$

In the zero-th approximation we determine two rectilinear normal modes (co- and anti-phase) corresponding to the two real roots of (4.2)

$$y_0 = 1.357\ x, \qquad y_0 = -0.926\ x \tag{4.4}$$

Adding rhe solution of the first approximation equation (4.3) in the form (2.2) and the solution of the zero-th approximation, we obtain the following expressions, respectively, for the vibration modes:

$$\begin{gathered} y = 1.345\ x + 0.003\ x^5 - 0.002\ x^7 + \dots \\ y = -0.962\ x - 0.002\ x^5 + 0.002\ x^7 + \dots \end{gathered} \tag{4.5}$$

Having available the vibrations mode (4.4) or (4.5), we can reduce the problem to the integration of a second order nonlinear equation of the form (1.7). The dimensionless periods of the vibrations for the co- and antiphasal modes are $T \approx 7.42$ and $T \approx 2.54$ in the zero-th approximation; taking account of the zero-th and first approximations $T \approx 6.71$ and $T \approx 1.88$. To estimate the accuracy of the asymptotic solution obtained, the system (4.1) was integrated numerically on an electronic digital computer. Variation of the initial conditions permitted extraction of two periodic solutions close to the normal vibration modes of the generating homogeneous system. The solutions obtained on the computer are characterized by the following parameters: $y / x \approx 1.349$ and $y / x \approx -0.950$ for $\dot{x} = 0$, $\dot{y} = 0$ and the periods of the vibrations are $T \approx 6.05$ and $T \approx 1.68$, respectively.

A comparison of the asymptotic solutions and the solutions computed on the computer shows that taking account of the zero-th and first approximations assures acceptable accuracy of the computation.

BIBLIOGRAPHY

1. Rosenberg, R. M. and Hsu, C. S., On the geometrization of normal vibrations of nonlinear systems. Analytical Methods of the Theory of Nonlinear Vibrations. Vol.1. Kiev, Acad. of Sci. UkrSSR Press, 1963
2. Manevich, L. I., Group-invariant solutions of the problem of normal vibrations. Hydro-aeromechanics and Elasticity Theory, №13, Dnepropetrovsk, 1971.
3. Liapunov, A. M., Collected Works, Vol. 2, Moscow, Akad. Nauk SSSR Press, 1956.
4. Malkin, I. G., Some Problems of the Theory of Nonlinear Vibrations, Moscow, Gostekhizdat, 1956.
5. Rosenberg, R. M. and Kuo, J. K., Nonsimilar normal mode vibrations of nonlinear systems having two degrees of freedom. J. Appl. Mech., Vol. 31, №2, 1964.
6. Pak, C. K. and Rosenberg, R. M., On the existence of normal mode vibrations of nonlinear systems. Quart. Appl. Math., Vol. 26, №3, 1968.
7. Cooke, C. H. and Struble, R. A., Perturbations of normal mode vibrations. Internat. J. Nonlinear Mech., Vol.1, №2, 1966.
8. Fikhtengol'ts, G. M., Course in Differential anf Integral Calculus, Vol. 2, Moscow, Fizmatgiz, 1962.

Translated by M.D.F.